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TWO PROBLEMS WITH MIXED BOUNDARY CONDITIONS FOR AN INCOMPRESSIBLE ISOTROPIC HYPERELASTIC MATERIAL*

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Within the framework of nonlinear elasticity theory there is considered the equilibrium of a layer of incompressible isotropic hyperelastic material under plane strain under the effect of gravity and forces P applied at infinity. The linearized equations generated by this state of stress and strain are investigated. It is shown that under for relationships between the material parameters, the layer thickness and the force P the equilibrium position can become unstable. Two problems are considered: the contact problem for a strip and the problem of a vertical crack of finite length emerging on the half-plane boundary. The action of the stamp and the crack is considered a small perturbation of the state of stress and strain caused by the action of the intrinsic weight and the force P.

1. Let x_1, x_2 be Cartesian coordinates of the undeformed state, and y_1, y_2 the Cartesian coordinates of the deformed state. The x_1 axis is along the upper boundary of the strip towards the right, and the x_2 axis is into the strip. Let the strip be subjected to its own weight and forces $P(P_1, P_2)$ applied at infinity; P_1 and P_2 are the respective projections on the axes x_1 and x_2. Then this state of stress and strain is described by the following system of equations (the equations of equilibrium, state, and the incompressibility condition, respectively):

$$\sigma_{ij,j} + \delta_{2i}\gamma^* = 0,\quad \sigma_{ij} AF_{ij} + pF_{ji}^{-1},\quad J = \det(F_{ij}) = 1;\quad A = 2\frac{dW}{dI} \tag{1.1}$$

For $x_2 = 0$ the boundary conditions are

$$\sigma_{12} = 0,\quad \sigma_{22} = 0 \tag{1.2}$$

Two kinds of boundary conditions corresponding to a smooth rigid base (problem A) and to rigid adhesion of the strip to the base (problem B) are considered on the lower boundary of the strip (i.e., $y_2 = h$ (problem A) and $y_1 = x_1, y_2 = h$ (problem B) for $\sigma_{12} = 0$.

$$\left(P_i = \int_0^h \sigma_{i1}\, dx_2,\quad i = 1, 2\right)$$

Here σ_{ij} is the Piola tensor, γ^* is the specific gravity, $F_{ij} = y_{i,j}$ is the strain gradient tensor, $I = F_{ij}F_{ij}$, $W(I)$ is the potential of the hyperelastic material, p is the hydrostatic pressure, F_{ji}^{-1} is the transpose tensor reciprocal to F_{ij}, δ_{ij} is the Kronecker delta, and h is the thickness of the strip.

We seek the solution in the form

$$y_1 = Rx_1 + \varphi(x_2),\quad y_2 = f(x_2),\quad R = \text{const} \tag{1.3}$$

We then obtain the following relations from the system (1.1) and (1.2):

$$A\varphi' = 0,\quad Af' + Rp = -\gamma^* x_2,\quad Rf' = 1 \tag{1.4}$$

From the first equation in (1.4) it follows that $\varphi' = 0$. For $A(I) \equiv 0$ the condition of ellipticity of the equilibrium equations is spoiled /1/, and of the Baker – Erickson inequality /2/. Because $\varphi' = 0$ a solution of the type (1.3) exists only for $P_2 = 0$, as we henceforth indeed assume.

Let us examine problem A. Taking account of (1.4), we have

$$p = R^{-1}(-\gamma^* x_2 - A_0 R^{-1}),\quad A_0 = A(I_0),\quad I_0 = R^2 + R^{-2}$$
$$(hA_0(R - R^{-3}) = R^{-2}\gamma^* h^2/2 + P_1)$$

*Prikl.Matem.Mekhan.,46,No.4,pp.700-704,1982 where R is determined from the equation in brackets. The boundary conditions of problem B determine $R=1$. The fixing of the infinitely remote point automatically determines the magnitude of the force $P_1=-\gamma^*h^2/2$ here.

2. Performing the standard linearization procedure in conformity with the method of small perturbations /2/, we obtain the following system of equations in dimensionless variables written in the coordinates of the deformed state:

$$\tau_{ij,j}=0,\quad u_{1,1}+u_{2,2}=0 \tag{2.1}$$

$$\tau_{11}=(Q+T+\gamma x_2)u_{1,1}+p,\quad \tau_{12}=(T+\gamma x_2)u_{2,1}+Tu_{1,2} \tag{2.2}$$

$$\tau_{22}=(Q+T+\gamma x_2)u_{2,2}+p,\quad \tau_{21}=(T+\gamma x_2)u_{1,2}+Eu_{2,1}$$

$$Q=R^2(1+m),\quad T=R^{-2},\quad E=R^2,\quad G=R^{-2}(1-m)$$

$$m=L(R^2-R^{-2})/A_0,\quad L=4d^2W/dI^2,\quad I=I_0,\quad \gamma=\gamma^*a/A_0$$

The dimensional variables (with the asterisk) are expressed as follows in terms of the dimensionless variables: the Piola tensor perturbation is $\tau_{ij}^*=A_0\tau_{ij}$, the displacement perturbation vector is $u_i^*=au_i$, the pressure perturbation is $p^*=A_0p$, the coordinates of the state of strain are $x_i^*=ax_i$, and a is a certain parameter with the dimensionality of a length. Substituting (2.2) into (2.1), we obtain

$$Qu_{1,11}+Tu_{1,22}+\theta_{,1}=0,\quad u_{1,1}+u_{2,2}=0 \tag{2.3}$$

$$Eu_{2,11}+Gu_{2,22}+\theta_{,2}=0,\quad \theta=p+\gamma u_2'$$

Applying the relationship obtained in /1/, the ellipticity condition for the system (2.3) can be represented in the form $G+Q>-2$. As is known /1,2/, buckling of the equilibrium position, the possibility of the appearance of solutions with weak discontinuities, is related to the loss of ellipticity.

3. We consider the action of a smooth stamp on the upper boundary of a heavy layer of incompressible hyperelastic material as a small perturbation of the state of stress and strain caused by the action of gravity and the forces P applied at infinity. We identify the parameter a from Sect.2 as the stamp half-width. Applying the Fourier transform in the variable x_2 to the system (2.3) and the corresponding boundary conditions, we obtain the following system

$$-\alpha^2Q\bar u_1+T\bar u_1''-i\alpha\bar\theta=0,\quad -i\alpha\bar u_1+\bar u_2'=0 \tag{3.1}$$

$$-\alpha^2E\bar u_2+G\bar u_2''+\bar\theta'=0,\quad \bar\theta=\bar p+\gamma u_2$$

$$\bar u_1'-i\alpha\bar u_2=0,\quad (G+T)\bar u_2'+p=\bar q,\quad x_2=0$$

$$\bar u_1'-i\alpha\bar u_2=\bar u_2=0,\quad x_2=\lambda \text{ (problem A)}$$

$$\bar u_1=\bar u_2=0,\quad x_2=\lambda \quad \text{(problem B)}$$

The prime denotes differentiation with respect to x_2 and the bar denotes the transform of the corresponding function, $q(x_1)$ is the contact pressure, and λ is the dimensionless thickness of the strip. Let $v(x_1)$ be the shape of the stamp. Solving the appropriate boundary value problems, we obtain integral equations for the contact pressure in the form

$$\pi v(x)=\int_{-1}^{1}q(t)K\left(\frac{x-t}{\lambda}\right)dt,\quad K(t)=\int_0^\infty L(u)\cos ut\,du \tag{3.2}$$

The form of $L(u)$ in problem A is determined by the roots η_1,η_2 of the equation

$$T\eta^2-(G+Q)\eta+E=0,\quad G+Q>-2 \tag{3.3}$$

If they are distinct $(\eta_1\neq\eta_2)$, then

$$L(u)=(\omega^2-\nu^2)((\omega^2-\nu^2)\gamma_0+Tu(d(\nu,\omega)\operatorname{cth}\nu u-d(\omega,\nu)\operatorname{cth}\omega u))^{-1} \tag{3.4}$$

$$\omega=\sqrt{\eta_1},\quad \nu=\sqrt{\eta_2},\quad d(a,b)=a(1+b^2)^2,\quad \gamma_0=\gamma\lambda$$

and if they are equal, then $\eta_1=\eta_2=\nu=\sqrt{E}$ and

$$L(u)=(\operatorname{ch}2u-1)/(Au\operatorname{sh}2u+Bu^2+\gamma_0(\operatorname{ch}2u-1)) \tag{3.5}$$

$$A=(3\nu+2\nu^{-1}-\nu^{-3})/2,\quad B=\nu+2\nu^{-1}+\nu^{-3}$$

If $d(\omega,\nu)>d(\nu,\omega)$ in (3.4), and $A<0$ in (3.5), then $L(u)$ has a poles on the real axis. This indicates instability of the prestress state.

For $d(\nu, \omega) = d(\omega, \nu)$ or $A = 0$ equation (3.2) is a Fredholm integral equation of the second kind and the strip behaves as a quasi-Winkler foundation, and as a pure Winkler foundation in the limit as $\lambda \to \infty$. For problem B

$$L(u) = u^{-1}(\operatorname{sh} 2u - 2u) / (2u^2 + \operatorname{ch} 2u + 1 + \gamma_0 (\operatorname{sh} 2u/2u - 1)) \quad (3.6)$$

It can be shown that for $L(u)$ representable by (3.4) for $d(\nu, \omega) > d(\omega, \nu)$ (3.5) for $A > 0$, and (3.6), the following asymptotic behavior is valid as $u \to \infty$:

$$L(u) \sim c_0 (1 + c_1 u^{-1} + c_2 u^{-2} + c_3 u^{-3} + O(u^{-4})); \quad c_i = \text{const} \quad (3.7)$$

and the asymptotic behavior of $L(u)$ as $u \to 0$ in problems A and B has the form, respectively

$$L(u) \sim c + O(u^2); \quad c = \text{const} \quad (3.8)$$

$$L(u) \sim cu^2 (1 + O(u^2)); \quad c = \text{const} \quad (3.9)$$

The representations (3.7)—(3.9) afford the possibility of using asymptotic methods for large and small lambda developed in /3,4/ to solve equations (3.2). Numerical computations performed in problem A for a material with a potential of the form $W = \mu / 2b ((1 + b/n (I - 2)^n - 1)$, $\mu, b > 0$ /5/, showed that the value λ_* at which these asymptotic methods join depends on the tension parameter R. For instance, for $b = 10, n = 2$, for $R = 0.8, \lambda_* = 2.8$ while for $R = 1.6, \lambda_* = 1.3$.

4. Let a narrow vertical shaft of length a now be carved in a heavy half-plane of an incompressible hyperelastic isotropic material. The shaft is reinforced by rigid horizontal braces that cancel the stress from the intrinsic weight acting on the vertical sides of the shaft. Such a narrow shaft will later be considered as a crack. The half-plane is considered the limit case of a layer adhering rigidly to the foundation. Without limiting the generality of the subsequent considerations, we consider the load applied only to the crack edges, where there are no tangential forces. The action of the load on the crack edges will be considered as a small perturbation of the state of stress and strain caused by the action of gravity. We identify the parameter a from Sect. 2 with the crack length. Then the system of equations and boundary conditions describing this state of stress and strain has the form

$$u_{i,11} + u_{i,22} + \theta_{,i} = 0, \quad u_{1,1} + u_{2,2} = 0; \quad \theta = p + \gamma u_2, \quad i = 1, 2 \quad (4.1)$$

$$\begin{aligned} &2u_{2,2} + \theta - \gamma u_2 = 0; \quad 0 \leqslant x_1 < \infty, \quad x_2 = 0 \\ &u_{1,2} + u_{2,1} = 0; \quad 0 \leqslant x_1 < \infty, \quad x_2 = 0 \end{aligned} \quad (4.2)$$

$$\begin{aligned} &u_{2,1} + (1 + \gamma x_2) u_{1,2} = 0, \quad x_1 = 0, \quad 0 \leqslant x_2 < \infty \\ &\theta - 2u_{2,2} - (\gamma x_2 u_2)_{,2} = 2f(x_2), \quad x_1 = 0, \quad 0 \leqslant x_2 < 1 \\ &u_1 = 0; \quad x_1 = 0, \quad 1 < x_2 < \infty \end{aligned} \quad (4.3)$$

We obtain from (4.1) that u_1 and u_2 are biharmonic functions, and θ is a harmonic function. We seek the biharmonic function u_2 in a special form (integration over α and β is everywhere performed later between 0 and ∞),

$$u_2(x_1, x_2) = \int B(\alpha)\alpha x_1 e^{-\alpha x_1} \cos \alpha x_2 d\alpha + \int K(\alpha)(1 - \alpha x_1) e^{-\alpha x_1} \sin \alpha x_2 d\alpha + \int (C(\beta) + \beta x_2 D(\beta)) e^{-\beta x_2} \sin \beta x_1 d\beta \quad (4.4)$$

We then have

$$\begin{aligned} u_1(x_1, x_2) = &- \int B(\alpha)(1 + \alpha x_1) e^{-\alpha x_1} \sin \alpha x_2 d\alpha - \\ &\int K(\alpha)\alpha x_1 e^{-\alpha x_1} \cos \alpha x_2 d\alpha + \int (C(\beta) - D(\beta) + \beta x_2 D(\beta)) e^{-\beta x_2} \cos \beta x_1 d\beta \\ \theta = 2 \{ &\int B(\alpha)\alpha e^{-\alpha x_1} \sin \alpha x_2 d\alpha + \\ &\int K(\alpha)\alpha e^{-\alpha x_1} \cos \alpha x_2 d\alpha - \int D(\beta)\beta e^{-\beta x_2} \cos \beta x_1 d\beta \} \end{aligned}$$

The boundary conditions for $x_2 = 0$ and $x_1 = 0$ become, respectively:

$$\begin{aligned} &\int \beta (D(\beta) - C(\beta)) \sin \beta x_1 d\beta = \int \alpha x_1 B(\alpha) e^{-\alpha x_1} d\alpha; \quad 0 \leqslant x_1 < \infty \\ &\int C(\beta)(\beta + \varkappa) \cos \beta x_1 d\beta = \int ((2 - \alpha x_1)\alpha K(\alpha) - \varkappa \alpha x_1 B(\alpha)) e^{-\alpha x_1} d\alpha; \\ &0 \leqslant x_1 < \infty \end{aligned} \quad (4.5)$$

$$\begin{aligned} &\int \alpha K(\alpha) \sin \alpha x_2 d\alpha + \varkappa x_2 \int \alpha B(\alpha) \cos \alpha x_2 d\alpha = 0; \quad 0 \leqslant x_2 < \infty \\ &\int \alpha B(\alpha) \sin \alpha x_2 d\alpha - \varkappa x_2 \int K(\alpha) d/dx_2 (x_2 \sin \alpha x_2) d\alpha = f(x_2) + \\ &\quad F(D, C); \quad 0 \leqslant x_2 < 1 \\ &\int B(\alpha) \sin \alpha x_2 d\alpha = 0; \quad 1 < x_2 < \infty \\ &F(D, C) = \int \{ D(\beta)\beta e^{-\beta x_2} + d/dx_2 ((C(\beta) + \beta x_2 D(\beta))(1 + \\ &\quad \varkappa x_2) e^{-\beta x_2} \} d\beta, \quad \varkappa = \gamma/2 \end{aligned} \quad (4.6)$$

Using the formulas for the Fourier sine and cosine transforms /6/ in (4.5) and (4.6), we obtain

$$\alpha K(\alpha) = \varkappa \frac{d}{d\alpha}(\alpha B(\alpha)) \tag{4.7}$$

$$\beta(D(\beta) - C(\beta)) = \frac{4}{\pi}\int B(\alpha)\alpha^3\beta(\alpha^2+\beta^2)^{-2}\,d\alpha$$

$$C(\beta)(\beta+\varkappa) = \frac{4}{\pi}\varkappa\int B(\alpha)\alpha(\alpha^2+\beta^2)^{-3}(3\alpha^2\beta^2-\beta^4)\,d\alpha$$

$$\int \alpha\left(B(\alpha) + \varkappa^2\frac{d}{d\alpha}\left(\alpha^{-1}\frac{d}{d\alpha}(\alpha B(\alpha))\right)\sin\alpha x_2\,d\alpha = \tag{4.8}$$

$$f(x_2) + \int B(\alpha)K(\alpha, x_2)\,d\alpha; \quad 0 \leqslant x_1 < 1$$

$$\int B(\alpha)\sin\alpha x_2\,d\alpha = 0; \quad 1 < x_2 < \infty$$

$$K(\alpha, u) = \frac{2}{\pi}\int \alpha e^{-\beta u}(\alpha^2+\beta^2)^{-3}(\beta+\varkappa)^{-1}\{\alpha^2\beta(\alpha^2+$$

$$\beta^2)(\beta+\varkappa)(1+\varkappa u)(2-\beta u) + \varkappa(3\alpha^2\beta^2-\beta^4)((1+\beta u)(\beta+\varkappa) -$$

$$\beta^2 u(2+\varkappa u))\}\,d\beta, \quad \delta = 1/\varkappa$$

We seek the solution of the system (4.8) in the form

$$B(\alpha) = \int_0^1 \varphi(t)\,\delta^2(\delta^2-t^2)^{-1}J(\alpha t)\,dt; \quad J(\alpha t) = \begin{cases} J_1(\alpha t), & \delta > 1 \\ J_1(\alpha t) - J_1(\alpha\delta), & \delta \leqslant 1 \end{cases} \tag{4.9}$$

where $J_1(z)$ is the Bessel function of first order. Therefore, the second equation of (4.8) can be satisfied automatically, and by differentiating under the integral sign, we have from the first equation

$$\int_0^\infty \alpha\sin\alpha x_2\,d\alpha\int_0^1 \varphi(t)J_1(\alpha t)\,dt = f(x_2) + \int_0^1 \varphi(t)S(x_2, t)\,dt \tag{4.10}$$

$$S(x_2, t) = \int_0^\infty K(\alpha, x_2)\,\delta^2(\delta^2-t^2)^{-1}J(\alpha t)\,d\alpha$$

Reducing (4,10) to a Fredholm equation of the second kind by a known method /7/, we obtain

$$\varphi(x_2) = F(x_2) + \int_0^1 R(x_2, t)\varphi(t)\,dt \tag{4.11}$$

$$F(x) = \frac{2}{\pi}\int_0^x f(t)\,t\,(x^2-t^2)^{-1/2}\,dt$$

$$R(x, t) = \frac{2}{\pi}\int_0^x S(t_1, t)\,t_1\,(x^2-x_1^2)^{-1/2}\,dt_1$$

$$R(x, t) = \frac{\delta^2}{\delta^2 - t^2}\int_0^\infty d\beta\left\{s\left[\varkappa x(sF(s))' - (sF(s))''\right]Q + \right.$$

$$\varkappa x\left[\varkappa\left[\Phi(s)(s^2+2) - sF(s) + 2\right] + \left(1 - \frac{2\beta}{\beta+\varkappa}\right)(sF(s) - \right.$$

$$\left.\left.\Phi(s) - 1) - \Phi(s))\right]T\right\}$$

$$z = \beta t, \quad s = \beta x, \quad u = \beta\delta; \quad F(z) = I_0(z) - L_0(z), \quad \Phi(z) = I_1(z) - L_1(z) - 2/\pi$$

$$Q = \begin{cases} (zF(z))'', & \delta > 1 \\ (zF(z))'' - (uF(u))'', & \delta \leqslant 1 \end{cases}, \quad T = \begin{cases} z(zF(z))', & \delta > 1 \\ z(zF(z))' - u(uF(u))', & \delta \leqslant 1 \end{cases}$$

Here I_i and L_i are, respectively, the modified Bessel and Struve functions. It can be shown that the norm of the operator defined by (4.11) is bounded in the space $C(0, 1)$ for any $\varkappa$ where the value $\varkappa_* < 1$ exists such that the operator (4.11) is a compression operator for $\varkappa < \varkappa_*$.

The coefficient of stress concentration at the angle of the crack is determined by the value of the function $\varphi(x)$ at the point $x = 1$. Numerical computations performed for $\varkappa < 1$ show that the magnitude of the stress intensity coefficient diminishes as δ grows.

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